Evasion differential game of one faster evader with bounded maneuverability from multiple pursuers

Gafurjan Ibragimov

4th International Seminar "Control Theory and the Theory of Generalized Solutions of Hamilton-Jacobi Equations" (CGS 2026), dedicated to the 80th anniversary of Academician A.I. Subbotin.

V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 100174, Tashkent, Uzbekistan.

January 27, 2026

Statement of the problem

In $\mathbb{R}^2$ the dynamics of pursuers x_i and evader y are given by the following equations

$$\begin{aligned} &\dot{x}_i = u_i,\ x_i(0) = x_{i0},\ i = 1, 2, ..., n, \\ &\dot{y} = v,\ y(0) = y_0, \end{aligned} \qquad (1)$$

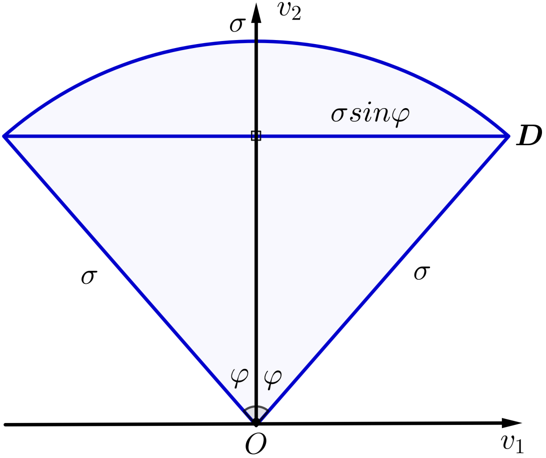
where

$x_i, x_{i0}, u_i, y, y_0, v \in \mathbb{R}^2$, $x_{i0} \neq y_0$,

$|u_i| \leq 1,\ i = 1, 2, ..., n,\ v \in D$,

u_i and v are control parameters of the pursuers and evader, respectively,

Statement of the problem



Definitions

$$D = \{(v_1, v_2) |\ v_1^2 + v_2^2 \leq \sigma^2,\ |v_1| \leq v_2 \tan\varphi,\ v_2 \geq 0\}$$

is the control set of the evader,
$\sigma > 1$ is a given number and $\varphi \in (0, \frac{\pi}{2})$ is a given angle.
Note that D is a sector of radius σ and central angle 2φ.
Also, D doesn't contain the control sets of the pursuers, the unit circle centered at the origin.

Admissible controls

Definition 1

Borel measurable functions
$u_i(t) = (u_{i1}(t), u_{i2}(t)),\ |u_i(t)| \leq 1,\ t \geq 0,$
and
$v(t) = (v_1(t), v_2(t)),\ v(t) \in D,\ t \geq 0,$
are called admissible controls of the pursuer x_i and evader y, respectively.

Let $H(0, \rho)$ be a circle of radius ρ and centered at the origin of $\mathbb{R}^2$. Let $\psi_1 = \psi_1(t),\ \psi_2 = \psi_2(t)$ be given functions.

Strategy of the evader

Definition 2

A function $V(t, \psi_1, \psi_2, y, x_1, ..., x_n, u_1, ..., u_n)$,

$$V : [0, \infty)^3 \times \mathbb{R}^{2(n+1)} \times H(0,1) \times \cdots \times H(0,1) \to D,$$

is called strategy of the evader, if, for any admissible controls $u_1 = u_1(t), ..., u_n = u_n(t)$ of pursuers and at $\psi_1 = \psi_1(t),\ \psi_2 = \psi_2(t)$, the following initial value problem

$$\begin{aligned}
&\dot{x}_1 = u_1,\ x_1(0) = x_{10},\\
&\vdots\\
&\dot{x}_n = u_n,\ x_n(0) = x_{n0},\\
&\dot{y} = V(t, \psi_1, \psi_2, y, x_1, ..., x_n, u_1, ..., u_n),\ y(0) = y_0,
\end{aligned}$$

has a unique solution $(x_1(t), ..., x_n(t), y(t))$, $t \geq 0$, with absolutely continuous components $x_1(t), ..., x_n(t)$, and $y(t)$.

Evasion is possible

Definition 3

We say that evasion is possible in game (1) if there exists a strategy V of the evader y such that for any admissible controls of the pursuers
$x_i(t) \neq y(t)$ for all $t \geq 0$ and $i = 1, ..., n$.

Problem 1

Find a condition to the angle φ and σ to guarantee that evasion is possible in game (1)*.*

Main result

Theorem 0.1

If $\sigma \sin \varphi > 1$*, then, regardless of the initial positions of the players, evasion is possible in game* (1)*.*

ε-approach time of the pursuer to the evader

$$\dot{x} = u,\ x(0) = x_0, \\ \dot{y} = v,\ y(0) = y_0, \tag{2}$$

Strategy for the evader:

$$V_0 = (0, \sigma),\ t \in [0, \theta), \tag{3}$$

where θ:

$$|y(\theta) - x(\theta)| = \varepsilon.$$

We call θ the ε-approach time of the pursuer to the evader.

The trajectory of the evader

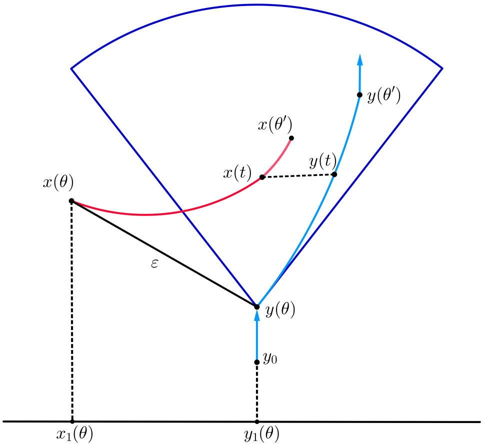


Figure: The trajectory of the evader when $x_1(\theta) \leq y_1(\theta)$

Manoeuvre

$$V(t) =$$

$$\begin{cases} (0, \sigma), & 0 \le t < \theta, \\ \left(\pm\left(\tau + |u_1(t)|\right), \sqrt{\sigma^2 - (\tau + |u_1(t)|)^2}\right), & \theta \le t < \theta', \\ (0, \sigma), & t \ge \theta', \end{cases} \quad (4)$$

where

$$\theta' = \theta + \frac{2\varepsilon}{\delta}, \quad \delta = \sqrt{(\sigma - 1)^2 - \tau^2}, \quad \tau = \sigma \sin \varphi - 1.$$

Lemma 4

Let the evader use strategy (4), *and* θ *be the* ε*-approach time of the pursuer* x *to the evader* y*. Then*

$$|y(t) - x(t)| \geq \varepsilon, \quad 0 \leq t \leq \theta, \tag{5}$$

$$|y(t) - x(t)| > \frac{\tau\varepsilon}{2\sigma}, \quad \theta \leq t \leq \theta', \tag{6}$$

$$y_2(t) - x_2(t) \geq \varepsilon, \quad t \geq \theta'. \tag{7}$$

Definitions of parameters

$$0 < \varepsilon_1 < \min_{i=1,\dots,n} |y_0 - x_{i0}|, \quad q = \frac{\tau\delta}{25\sigma^2},$$

$$\varepsilon_{k+1} = q \cdot \varepsilon_k, \; k = 1, 2, \cdots .$$

$t = \theta_1 > 0$ is the ε_1-approach time of a pursuer x_{i_1} to the evader if

$$|x_{i_1}(\theta_1) - y(\theta_1)| = \varepsilon_1$$

and

$$|x_i(t) - y(t)| > \varepsilon_1, \quad 0 \le t < \theta_1, \quad i = 1, 2, \dots, n.$$

The ε_k-approach time

Let θ_{k-1}, $k \geq 2$, be the ε_{k-1}-approach time.

Then we define the ε_k-approach time $\theta_k > \theta_{k-1}$ as the time when for a pursuer x_{i_k} the following conditions are satisfied

$$|y(\theta_k) - x_{i_k}(\theta_k)| = \varepsilon_k,$$

and

$$|y(t) - x_i(t)| > \varepsilon_k, \quad 0 \leq t < \theta_k, \quad i = 1, ..., n.$$

θ_k can be the ε_k-approach time for several x_i:

θ_k is the ε_k-approach time for several pursuers

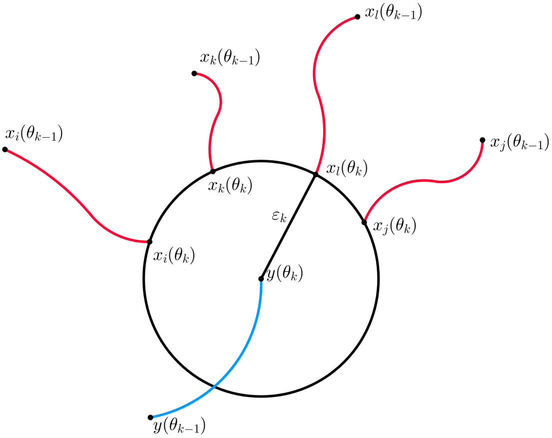


Figure: θ_k is the ε_k-approach time for x_i, x_j, x_k, x_l.

Manoeuvres for the evader

Let $y_0 = (0, 0)$ and

$$\theta'_k = \theta_k + \frac{2\varepsilon_k}{\delta},\ k = 1, 2, ..., \quad \theta_0 = 0,\ \theta'_0 = \infty.$$

$\theta'_1, \theta'_2, \dots$ is not necessarily monotone. The manoeuvre of the evader against the pursuer $x_k, k = 1, 2, ...,$ as follows

$$V_{k1}(t) = \begin{cases} \tau + |u_{k1}(t)|, & x_{k1}(\theta_k) \le y_1(\theta_k), \\ -(\tau + |u_{k1}(t)|), & x_{k1}(\theta_k) > y_1(\theta_k), \end{cases} \qquad (8)$$

$$V_{k2}(t) = \sqrt{\sigma^2 - V_{k1}^2(t)}.$$

Evasion strategy

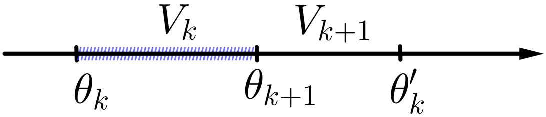
First, the evader moves starting from the time $\theta_0 = 0$ along the y-axis with $v(t) = V_0 = (0, \sigma)$.
If the ε_1-approach time never occurs

$$|y(t) - x_i(t)| > \varepsilon_1, \quad i = 1, 2, ..., n, \quad t \geq 0,$$

Hence, $x_i(t) \neq y(t)$, $t \geq 0$, $i = 1, 2, ..., n$.
Let the ε_1-approach time $\theta_1 > 0$ occur.
Suppose θ_k, $k \geq 1$, happened. Then, depending on occurrence of the time (unspecified) θ_{k+1}, the evader's strategy is constructed as follows:

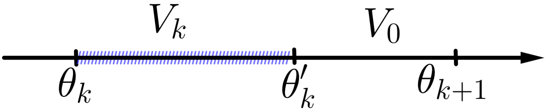
Evasion strategy

(i) $v(t) = V_k(t)$ on $[\theta_k, \theta_{k+1})$, if $\theta_{k+1} \in [\theta_k, \theta'_k)$.



$(5a)$

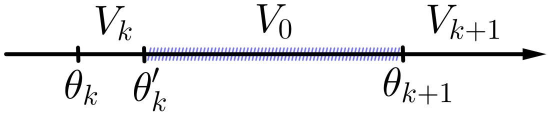
(ii) $v(t) = V_k(t)$ on $[\theta_k, \theta'_k)$, if $\theta_{k+1} \notin [\theta_k, \theta'_k)$.



$(5b)$

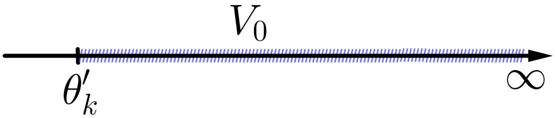
Evasion strategy

(iii) $v(t) = V_0$ on $[\theta'_k, \theta_{k+1})$, if $\theta_{k+1} \in [\theta'_k, \infty)$.



$(5c)$

(iv) $v(t) = V_0$ on $[\theta'_k, \infty)$, if θ_{k+1} never occurs.



$(5d)$

The functions $\psi_1(t)$ and $\psi_2(t)$

Let $\psi_1(t) = \max_{\theta_k \le t} \theta_k = \theta_K$ for some $K \ge 0$. That is, θ_K represents the greatest value among the numbers θ_k that do not exceed the current time t.

Then, we define $\psi_2(t) = \theta'_K$. Note that $K = K(t)$ depends on t and is redefined at each time θ_k.

Further the intervals $\theta_K \le t < \theta'_K$ and $t \ge \theta'_K$ are of importance to assign a manoeuvre for the evader.

It should be noted that these intervals are updated at each time θ_k too.

The functions $\psi_1(t)$ and $\psi_2(t)$

The functions $\psi_1(t)$ and $\psi_2(t)$ are used to assign a manoeuvre $V_K(t)$ for the evader at the current time t. The evader's strategy defined by items (i)–(iii) means that $v(t) = V_K(t)$ for current time t whenever $\theta_K \leq t < \theta'_K$, and $v(t) = V_0$ if $t \geq \theta'_K$.

For example, $K = K(t) = 0$, $\theta_0 \leq t < \theta_1$.
Hence, $v(t) = V_K(t) = V_0$, $\theta_0 \leq t < \theta_1$.
Next, $K = K(t) = 1$ for $\theta_1 \leq t < \theta_2$, and so $v(t) = V_1(t)$ whenever $\theta_1 \leq t < \theta'_1$ and $K(t) = 1$, and $v(t) = V_0$ whenever $t \geq \theta'_1$ and $K(t) = 1$.

Evasion is possible

Note that the strategy constructed above differs from the strategy of Chernous'ko [2] in several essential aspects.

In [2], when the evader moves along a spiral L_j, it preserves a constant distance from the corresponding pursuer, whereas under our strategy the evader does not maintain a fixed distance from the pursuers.

Evasion is possible

Moreover, in [2], after completing the j-th spiral L_j, the evader continues its motion along the arcs of the spirals $L_j, L_{j-1}, \ldots, L_1$ or along a portion of the horizontal axis,

while in the present work the evader switches from the manoeuvre $V_k(t)$ to $V_{k+1}(t)$ or to V_0 without retracing previously visited arcs.

Evasion is possible

Another fundamental difference concerns the number of manoeuvres: in [2] this number is estimated by $2^m - 1$, whereas in the present work it is bounded by $m(m+1)/2$, reducing exponential growth to quadratic.

Finally, in [2] the control set of the evader contains the control sets of the pursuers, while in the present work the evaderâĂŹs control set D does not contain the pursuersâĂŹ control sets $H(0, 1)$.

Evasion is possible

1. Azamov A., Ibragimov G., Kurbanov A. Evasion differential game of one faster evader with bounded maneuverability from multiple pursuers. Dynamic Games and Applications. 2025. DOI: 10.1007/s13235-025-00684-w

2. Chernous'ko F.L. A Problem of Evasion of Several Pursuers. Prikl. Mat. Mekh., 1976, vol. 40(1), pp. 14–24.

Evasion is possible

3. Ibragimov G.I. Evasion differential game of one evader and many slow pursuers. Dynamic Games and Applications. 2024. Vol. 14, No. 3, pp. 665–685. DOI: 10.1007/s13235-023-00501-2

4. Ibragimov G., Alias I.A., Syed Mafdzot Sh.A. Evasion differential game with many pursuers versus one evader whose control set is a sector. ScienceAsia. 2017. Vol. 43, No. 2, pp. 115–122. DOI: 10.2306/scienceasia1513-1874.2017.43.115

Evasion is possible

5. Isaacs R. Differential games. John Wiley & Sons, New York, NY, USA, 1965.

6. Krasovskii N.N., Subbotin A.I. Game-Theoretical Control Problems. Springer, New York, NY, 1988.

7. Petrosyan L.A. Differential Games of Pursuit. World Scientific, Singapore, 1993.

8. Pontryagin L.S. Izbrannye Trudy; MAKS Press: Moscow, 2004.