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A quasi-linear pursuit game for neutral-type differential-difference equations

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Abstract

This paper studies quasilinear differential games described by a system of differential-difference equations of neutral type under geometric constraints on the players' controls. Modifications of the third pursuit method for differential-difference equations of neutral type are developed. New sufficient conditions on the process parameters for game completion in a certain finite time are obtained.

1. Statement of the problem

A quasi-linear pursuit differential game described by a system of neutral-type differential-difference equations [5] is considered

$$\dot{z}(t) = Az(t) + \sum_{i=0}^{m} B_i \dot{z}(t-h_i) + \sum_{i=0}^{m} C_i z(t-h_i) - f(u,v), \ t \geq 0, \tag{1}$$

where $z(t) = (z_1(t), \cdots, z_n(t))$ is the vector of phase coordinates in the space $\mathbb{R}^n$, $n \geq 1; A, B_i \, (i = 0, 1, \cdots, m), C_i \, (i = 0, 1, \cdots, m)$ are constant square matrices of dimension $(n \times n)$; $0 = h_0 < h_1 < \cdots < h_m$ are constants; u and v are the control parameters of the pursuer and the evader, respectively, $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$, $p \geq 1$, $q \geq 1$. The parameters u and v are chosen as measurable vector functions $u = u(\cdot)$ and $v = v(\cdot)$ satisfying the geometric constraints

$$u \in P, \quad v \in Q, \ \ 0 \leq t < +\infty, \tag{2}$$

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where P and Q are nonempty compact subsets of the spaces $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. The function $f(u, v)$, $f : P \times Q \to \mathbb{R}^n$, is continuous in all its variables the control block. In addition, a terminal set is specified in the space $\mathbb{R}^n$, which is denoted by the symbol M.

The initial position for the pursuit problem (1) is an n-dimensional absolutely continuous function $\varphi(t)$ defined on the interval $[-h_m, 0]$, that is,

$$\Big\{\varphi(\cdot) : z(t) = \varphi(t),\ t \in [-h_m, 0],\ z(0) = \varphi(0) \in \mathbb{R}^n \setminus M\Big\}. \quad (3)$$

Measurable functions $u = u(t), 0 \leq t < +\infty$ and $v = v(t), 0 \leq t < +\infty$ that satisfy the constraints (2) are called the *admissible controls* of the pursuer and the evader, respectively.

Let $u = u(t), 0 \leq t < +\infty$ and $v = v(t), 0 \leq t < +\infty$ be admissible controls of the players in the game (1), (2). To the controls $u(\cdot)$, $v(\cdot)$ and the initial position $\varphi(\cdot)$ (see (3)) corresponds a solution $z(t) = z(\varphi(\cdot), u(\cdot), v(\cdot))$ of equation (1), which at $t = 0$ starts from the initial position $\varphi(\cdot)$.

The state of the system (1) at the current time t, $t \in [0, +\infty)$, is defined by the vector-function $z^{'}(\cdot)$, $z^{'}(\cdot) = (z^{'}_1(\cdot), \cdots, z^{'}_n(\cdot))$ where $z^{'}(s) = \{z(t + s), -h_m \leq s \leq 0\}$ which represents a segment of the realized trajectory of the system (1) on the time interval $[t - h_m, t]$.

By choosing their controls as measurable vector functions $u = u(\cdot)$ and $v = v(\cdot)$ satisfying the constraints (2), each player influences equation (1) in pursuit of their own objectives. The pursuit begins at $t = 0$ from the initial position $\varphi(\cdot)$ and is considered completed at the moment $\tilde{t} = \tilde{t}[\varphi(\cdot)]$, when the phase point $z(t)$ first reaches the terminal set M, i.e., $z(\tilde{t}) \in M$. The goal of the evader is to delay the end of the game as much as possible.

2. Auxiliary constructions.

Throughout the following: a) The terminal set M is cylindrical and has the form $M = M_0 + M_1$, where M_0 is a linear subspace of $\mathbb{R}^n$ and M_1 is a convex compact subset of the subspace L, where L is the orthogonal complement of M_0 in $\mathbb{R}^n$ (i.e., $M_0 \oplus L = \mathbb{R}^n$); b) Denote by π the matrix of the orthogonal projection operator from $\mathbb{R}^n$ onto $L : \pi : \mathbb{R}^n \to L$; c) The integral of a single-valued or multi-valued function (or multi-valued mapping) under the integral is understood as its Lebesgue integral [2].

Let S denote the set of points of the form $S = \Big\{t : t = \sum_{i=0}^{m} j_i h_i,\ j_i - integer\Big\}$, and let S^0 denote the intersection of S with the half-plane $(0, \infty)$ i.e. $S^0 = S \bigcap \Big(0, \infty\Big)$.

Let τ be a positive number and $t \in [0, \tau]$. Let the admissible controls $u = u(t)$, $v = v(t)$ be chosen on the interval $[0, \tau]$, $\tau > 0$. Then for the solution $z(\tau)$ of equation (1) with the initial condition (3), the following representation holds [1]:

$$z(\tau) = \Big[K(\tau) - \sum_{i=0}^{m} K(\tau - h_i) B_i\Big]\varphi(0) +$$

$$+ \sum_{i=0}^{m} \int_{-h_i}^{0} K(\tau - t - h_i)\Big[B_i \dot\varphi(t) + C_i \varphi(t)\Big] dt -$$

$$- \int_{0}^{\tau} K(\tau - t) f(u(t), v(t)) dt, \tag{4}$$

where $K(t), -\infty < t \le \tau$, is the unique matrix function possessing the following properties [1,3]:

a) $K(t) = \tilde 0,\ t < 0$, $\tilde 0$ is the zero matrix of order n;
b) $K(0) = E$, where E is the identity matrix of order n;
c) the function $\sum_{i=0}^{m} C_i K(t - h_i)$ is continuous on $[0, +\infty)$;
d) $K(t)$ satisfies the matrix differential equation

$$\dot K(t) = AK(t) + \sum_{i=0}^{m} B_i \dot K(t - h_i) + \sum_{i=0}^{m} C_i K(t - h_i), \text{ at } t > 0,\ t \notin S^0. \tag{5}$$

The existence and uniqueness of the matrix function $K(t), -\infty < t \le \tau$, satisfying conditions (a)-(d) can be proved by the usual method of stepwise successive integration of equation (5). The function $K(t)$ belongs to the class C^1 for C^1 at $t > 0,\ t \notin S^0$, but in the general case it has first-kind discontinuities at the points of the set S^0.

Definition 1. We shall say that in the game (1), (2) it is possible to complete the pursuit from the initial position $\varphi(\cdot)$ within the time $T = T(\varphi(\cdot)), 0 \le T < +\infty$, if there exists a function $u(t, v)$, $0 \le t \le T(\varphi(\cdot))$, $v \in Q$, taking values in the set P, such that for any measurable function $v = v(t)$, $0 \le t \le T(\varphi(\cdot))$, $v(t) \in Q$, the function $u(t, v(t))$, $0 \le t \le T(\varphi(\cdot))$, is measurable, and the corresponding trajectory $z = z(t)$, i.e., the solution of equation (1) with the initial condition (3), at some moment $t = t^* \in [0, T]$ reaches the terminal set M, i.e., $z(t^*) \in M$.

In conducting the pursuit, the pursuer needs to know information about the objects. It is assumed that the pursuer knows, at each moment of time $t \geq 0$, the solution $z(s)$ on the interval $-h_m \leq s \leq t$ and the control $v(s)$, $0 \leq s \leq t$.

The number $T(\varphi(\cdot))$ is called the pursuit time from the point $\varphi(\cdot) \in X$, and the function $u(t, v)$, $0 \leq t \leq T(\varphi(\cdot))$, $v \in Q$, is called the pursuit function.

It is required to find the initial positions $\varphi(\cdot)$ from which, in the game (1), the pursuit can be completed in a finite time $T = T(\varphi(\cdot))$.

Let $M(t)$, $0 \leq t \leq \tau$, be an arbitrary compact-valued multi-valued mapping satisfying the condition [2, 3]

$$\int_0^\tau M(t)dt \subset M_1. \tag{6}$$

Now we formulate a sufficient condition for the possibility of completing the pursuit in the game (1), (2). The completion of the pursuit is understood in the sense of the definition introduced above.

Consider the multi-valued mappings

$$\pi K(t) f(P, v) = \{\pi K(t) f(u, v) : u \in P\}, \text{ for } v \in Q.$$

Define

$$\hat{W}(M(t), t) = \bigcap_{v \in Q} \Big[M(t) + \pi K(t) f(P, v) \Big], \ 0 \leq t \leq \tau, \tag{7}$$

where the intersection is taken over all $v \in Q$.

Assumption 1. *There exists a multi-valued mapping $M(t)$, $0 \leq t \leq \tau$, such that the set $\hat{W}(M(t), t)$, $t \in [0, \tau]$ is nonempty for all $t \in [0, \tau]$.*

By virtue of this assumption and the properties of the multi-valued mapping $\hat{W}(M(t), t)$, there exists at least one Borel measurable summable selector in it, which allows us to introduce the Lebesgue integral [2] of the multi-valued mapping $\hat{W}(M(t), t)$, i.e.,

$$W(\tau) = \bigcup_{M(\cdot)} \int_0^{\tau} \hat{W}(M(t), t)dt, \; W(0) = M_1. \tag{8}$$

Let, for $\tau \geq 0$,

$$\Phi(\tau)\varphi(\cdot) = \Big[K(\tau) - \sum_{i=0}^{m} K(\tau - h_i)B_i\Big]\varphi(0)$$

$$+ \sum_{i=0}^{m} \int_{-h_i}^{0} K(\tau - t - h_i)\Big[B_i\dot{\varphi}(t) + C_i\varphi(t)\Big]dt.$$

According to (8), we introduce a function that defines the minimal guaranteed capture time of the evader by the pursuer for the modification of the third method of the pursuit problem, and set [4]

$$T(\varphi(\cdot)) = \inf\{\tau \geq 0 : \pi\Phi(\tau)\varphi(\cdot) \in W(\tau)\}.$$

Theorem 1. *Let the assumption hold and $T(\varphi(\cdot)) \neq \emptyset$. In addition, there exists a positive number $\tau = \tau_1(\varphi(\cdot))$, $\tau_1(\varphi(\cdot)) \in T(\varphi(\cdot))$, such that the inclusion*

$$\pi\Phi(\tau_1)\varphi(\cdot) \in W(\tau_1). \tag{9}$$

holds. Then, in the game (1), (2), *it is possible to complete the pursuit from the given initial position $\varphi(\cdot)$ within the finite time $T = \tau_1(\varphi(\cdot))$.*

Proof. According to (9) and (8), there exists a measurable closed-valued multi-valued mapping $M(t)$, $0 \leq t \leq \tau_1$, such that

$$\int_0^{\tau_1} M(t)dt \subset M_1,$$

and

$$\Phi(\tau_1)\varphi(\cdot) \in \hat{W}(M(\cdot), \tau_1). \tag{10}$$

By virtue of inclusion (10) and in accordance with the definition of the integral of a multi-valued mapping (8), there exists a summable function $\tilde{w}(t)$, $0 \leq t \leq \tau_1$, for which the following equality holds:

$$\int_0^{\tau_1} \tilde{w}(t)dt = \Phi(\tau_1)\varphi(\cdot), \quad \tilde{w}(t) \in \hat{W}(M(t), t), \;\; 0 \leq t \leq \tau_1. \tag{11}$$

Therefore,

$$\tilde{w}(\tau_1 - t) \in \bigcap_{v \in Q} \Big\{ \Big[M(\tau_1 - t) + \pi K(\tau_1 - t) f(P, v) \Big] \Big\},$$

for some $t \in [0, \tau_1]$.

Consequently (see (11)), the equation

$$\tilde{w}(\tau_1 - t) = m + \pi K(\tau_1 - t) f(u, v), \tag{12}$$

with respect to the unknown pair $(u, m) \in P \times M(\tau_1 - t)$, for each fixed pair $(t, v) \subset [0, \tau_1] \times Q$, has a solution. It can be shown that there exists a unique solution $(u(t, v), m(t))$ of equation (12), where the component $u(t, v)$ is the *lexicographically minimal* among all solutions (u, m) of equation (12) (the proof uses, in particular, the closedness of the set M).

Further, similarly to the Filippov-Casten lemma [5], it is established that for an arbitrary measurable function $v(t)$, $t \in [0, \tau_1]$, $v(t) \in Q$, the function $u(t, v(t))$, $t \in [0, \tau_1]$, is measurable. The summability of the function $(m(t, v(t)))$, $t \in [0, \tau_1]$, follows from equation (12).

It is asserted that if the side controlling the parameter u chooses the function $u(t, v)$, $t \in [0, \tau_1]$, $v \in Q$, while the side controlling v chooses an arbitrary function, then the trajectory $z(t)$, $0 \le t \le \tau_1$, of equation (1) with the initial condition (3) reaches the set M, i.e., the inclusion $z(t) \in M$ will hold at some $t = t^* \in [0, \tau_1]$.

Indeed, suppose, contrary to the assertion, that $z(t) \notin M$ on the interval $[0, \tau_1]$. Then, from the Cauchy formula [4] for the solution representation (after projection onto L) of equation (1) at $\tau = \tau_1$, we obtain

$$\pi z(\tau_1) = \Big[\pi K(\tau_1) - \sum_{i=0}^{m} \pi K(\tau_1 - h_i) B_i \Big] \varphi(0) +$$

$$+ \sum_{i=0}^{m} \int_{-h_i}^{0} \pi K(\tau_1 - t - h_i) \Big[B_i \dot{\varphi}(t) + C_i \varphi(t) \Big] dt -$$

$$- \int_{0}^{\tau_1} \pi K(\tau_1 - t) f(u(t, v(t)), v(t)) dt. \tag{13}$$

Taking into account the control selection law (13) and the equality in (11), from the formula above we get

$$\pi z(\tau_1) = \Phi(\tau_1)\varphi(\cdot) - \int_0^{\tau_1} \tilde{w}(\tau_1 - t)dt + \int_0^{\tau_1} m(\tau_1 - t)dt,$$

and since $m(t) \in M$ on the interval $[0, \tau_1]$, we have

$$\pi z(\tau_1) = \int_0^{\tau_1} m(t)dt \in \int_0^{\tau_1} M(t)dt \subset M_1.$$

Therefore, $\pi z(\tau_1) \in M_1$, which is equivalent to $z(\tau_1) \in M$, contrary to our assumption.

The proof of Theorem is completed.

3. References

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